Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(99)00078-7
J. Appl. Maths Mechs, Vol. 63, No. 4, pp. 633–640, 1999

0021-8928/99/$—see front matter

THE STIFFNESSES OF NON-HOMOGENEOUS PLATES†

A. G. KOLPAKOV and I. G. SHEREMET

Novosibirsk

(*Received* 30 *May* 1997)

Variational principles and estimates are obtained for the stiffnesses of non-homogeneous plates of periodic structure, which arise from the asymptotic method of analysing the problem of the theory of elasticity in regions of small thickness [1].

1. FORMULATION OF THE PROBLEM

Consider an elastic solid of periodic structure, which occupies a region of small thickness ε (Fig. 1). The characteristic size of a periodicity cell P_ε of the solid in the Ox_1x_2 plane is also of the order of ε. The elasticity constants a_{ijkl} $(\mathbf{x}/\varepsilon)$ of the solid are functions of the argument $\mathbf{x}$, and are periodic in x_1, $x_2 \in P_\varepsilon$.

We know [1] (for the case of a flat plate [2]), that as $\varepsilon \to 0$ the solution of the problem of the theory of elasticity for the solid described tends to the solution of the plate-theory problem with the governing equations

$$N_{\alpha\beta} = A^0_{\alpha\beta\gamma\delta}e_{\gamma\delta} + A^1_{\alpha\beta\gamma\delta}\rho_{\gamma\delta}$$
$$M_{\alpha\beta} = A^1_{\alpha\beta\gamma\delta}e_{\gamma\delta} + A^2_{\alpha\beta\gamma\delta}\rho_{\gamma\delta};\ \alpha,\ \beta,\ \gamma,\ \delta = 1, 2 \tag{1.1}$$

where $N_{\alpha\beta}$ and $M_{\alpha\beta}$ are the forces and moments, $e_{\gamma\delta}$, $\rho_{\gamma\delta}$ are the strains in the plane of the plate and the curvatures, and $A^{\nu+\mu}_{\alpha\beta\gamma\delta}$ $(\nu, \mu = 0.1)$ are the stiffnesses of the plate (when $\nu = \mu = 1$ they are flexural and when $\nu = \mu = 0$ they are in the plane of the plate), calculated as follows [1, 2]. The so-called cell problem is solved, namely

$$(a_{ijkl}(\mathbf{y})\, N^{\gamma\delta\nu}_{k,l} + (-1)^\nu\, a_{ij\gamma\delta}(\mathbf{y})y_3^\nu)_{,j} = 0 \text{ in } P_1$$
$$(a_{ijkl}(\mathbf{y})\, N^{\gamma\delta\nu}_{k,l} + (-1)^\nu\, a_{ij\gamma\delta}(\mathbf{y})y_3^\nu,\ n_j = 0 \text{ on } \Gamma_1 \tag{1.2}$$

where the function $\mathbf{N}^{\gamma\delta\nu}(\mathbf{y})$ is periodic in $y_1, y_2 \in P_1 = \partial/\partial y_j)$. The periodicity condition has the following meaning: Γ_1 is the free surface of the periodicity cell. The remaining part of the boundary of the periodicity cell, which we denote by Γ_0, is the contact surfaces of the neighbouring periodicity cells. It is to these surfaces that the periodicity condition applies.

We will calculate the stiffness from the formula

$$A^{\nu+\mu}_{\alpha\beta\gamma\delta} = \left\langle (-1)^\mu y_3^\mu (a_{\alpha\beta kl}(\mathbf{y}) N^{\gamma\delta\nu}_{k,l} + (-1)^\nu y_3^\nu (a_{\alpha\beta\gamma\delta}(\mathbf{y}))\right\rangle \tag{1.3}$$

($\langle\cdot\rangle = 1/mesS_1 \int_{P_1} \cdot\, \mathbf{dy}$ is the average of $P_1 = \varepsilon^{-1}P_\varepsilon = \{\mathbf{y} = \mathbf{x}/\varepsilon : \mathbf{x} \in P_\varepsilon\}$ over the periodicity cell in dimensionless coordinates $\mathbf{y} = \mathbf{x}/\varepsilon$ and S_1 is the projection of P_1 onto the Oy_1y_2 plane).

We will concentrate our investigation on the stiffnesses and will pay particular attention to the flexural stiffnesses.

A formula was derived in [2, formula (6.27)] for flat plates, which can be written as follows:

$$A^2_{\alpha\beta\gamma\delta} = \left\langle a_{ijkl}(\mathbf{y})(N^{\gamma\delta 1}_{k,l} - y_3\delta_{k\gamma}\delta_{l\delta})(N^{\alpha\beta 1}_{i,j} - y_3\delta_{i\alpha}\delta_{j\beta})\right\rangle \tag{1.4}$$

($\delta_{ii} = 1$ and $\delta_{ij} = 0$ when $i \neq j$). It represents $A^2{}_{\alpha\beta\gamma\delta}$ in the form of a quadratic functional, which is convenient for obtaining variational principles.

We will verify that formula (1.4) remains true for plates with non-flat surfaces (for example, for ribbed, corrugated, etc. surfaces). To do this we will show that (1.4) is a corollary of relations (1.2) and (1.3).

†*Prikl. Mat. Mekh.* Vol. 63, No. 4, pp. 662–670, 1999.

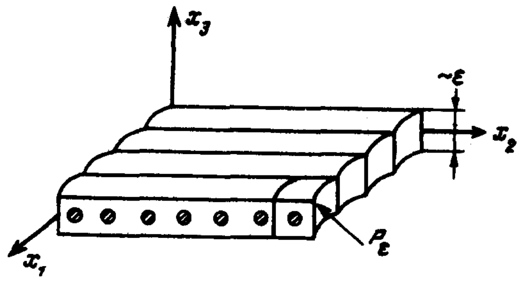

Fig. 1.

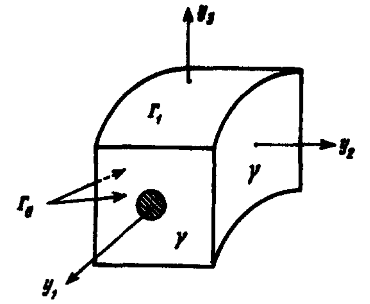

Fig. 2.

We multiply the equation from (1.2) by $N_i^{\alpha\beta1}$ and integrate the result by parts on P_1, taking the remaining conditions from (1.2) into account. We obtain the equation

$$0=\left\langle a_{ijkl}(\mathbf{y})(N^{\gamma\delta1}_{k,l}-y_3\delta_{k\gamma}\delta_{l\delta})N^{\alpha\beta1}_{i,j}\right\rangle \tag{1.5}$$

Subtracting this equation from (1.3) with $\nu = \mu = 1$ we obtain (1.4).

2 VARIATIONAL PRINCIPLES AND ESTIMATES FOR FLEXURAL STIFFNESSES

To obtain the variational principles for stiffnesses we will establish a relation between the cell problem (1.2), its Lagrange and Castigliano functionals [3, 4] and the stiffnesses (1.4) (the prototype of this method has been described previously for beams in [5]). We set up the Lagrange functional $J_u(\mathbf{u})$ for problem (1.2). It has the form [3, 4]

$$J_u(\mathbf{u})=\left\langle a_{ij\gamma\delta}(\mathbf{y})y_3u_{i,j}\right\rangle-\tfrac{1}{2}\left\langle a_{ijkl}(\mathbf{y})u_{i,j}u_{k,l}\right\rangle \tag{2.1}$$

and is considered on the set of possible displacements

$$V=\{\mathbf{u}(\mathbf{y})\in H^1(P_1)\text{: }\mathbf{u}(\mathbf{y})\text{ is periodic in }y_1,y_2\in P_1\} \tag{2.2}$$

We can obtain the Castigliano functional by considering the problem that is the dual of the problem of maximizing $J_u(\mathbf{u})$ on V. The functional of the dual problem can be calculated in the same way as that described previously [4, Chapter 7, Section 4] with a single change due to the condition of periodicity in definition (2.2) (it changes the condition of rigid clamping from [4]).

For the case considered (using the notation used previously [4]) we will put $f_i = -(a_{ij\gamma\delta}(\mathbf{y})\,y_3)_{,j}$ in P_1—the mass forces and $g_i = a_{ij\gamma\delta}(\mathbf{y})\,y_3n_j$ on Γ_1—the surface stresses. We now calculate $F^*(-\Lambda^*\sigma)$ in the same way as previously [4, Chapter 7, Section 4] and we obtain the equations

$$\sigma_{ij,j}+f_i=0\text{ in }P_1,\ \sigma_{ij}n_j-g_i=0\ \text{ on }\Gamma_1 \tag{2.3}$$

For functions which satisfy conditions (2.3) we have $\Lambda\boldsymbol{v} = (v_{i,j} + v_{j,i})/2$—the strains. Here $\boldsymbol{v} \in V$ and $\boldsymbol{v} = 0$ on Γ_1. For smooth functions

$$\left\langle\sigma_{ij}n_j,v_i\right\rangle=\int_\gamma[\sigma_{ij}n_j]v_i\,dy \tag{2.4}$$

The square brackets denote the difference in the values of the function on opposite faces of the periodicity cell (Fig. 2). When obtaining formula (2.4) we took into account the fact that the values of the function $\boldsymbol{v} \in V$ are equal on these faces. From (2.3) we obtain, in the notation used previously in [4, Chapter 7, Section 4]

$$F^*(-\Lambda^*\sigma)=\begin{cases}0, \text{ if } (\sigma_{ij}-a_{ij\gamma\delta}(\mathbf{y})y_3)_{,j}=0\text{ in }P_1\\ (\sigma_{ij}-a_{ij\gamma\delta}(\mathbf{y})y_3)n_j=0\text{ on }\Gamma_1\\ [\sigma_{ij}n_j]=0\text{ on }\gamma\\ +\infty\text{ in the remaining cases}\end{cases} \tag{2.5}$$

The last equation in (2.5) means the periodicity of $\sigma_{ij}n_j$ in $y_1, y_2 \in P_1$.

Remark 1. The elasticity constants $a_{ijkl}(y)$ are periodic in $y_1, y_2 \in P_1$, and hence the periodicity conditions $(\sigma_{ij} - a_{ij\gamma\delta}(y)\, y_3)n_j$ and $\sigma_{ij}n_j$ in $y_1, y_2 \in P_1$ are equivalent.

We will introduce the set of permissible stresses

$$\Sigma = \{\sigma_{ij} \in L(P_1) : (\sigma_{ij} - a_{ij\gamma\delta}(\mathbf{y})y_3)_{,j} = 0 \text{ in } P_1,\ (\sigma_{ij} - a_{ij\gamma\delta}(\mathbf{y})y_3)n_j = 0 \text{ on } \Gamma_1,$$
$$(\sigma_{ij} - a_{ij\gamma\delta}(\mathbf{y})y_3)n_j \text{ are periodic in } y_1, y_2 \in P_1\} \tag{2.6}$$

The Castigliano functional $J_\sigma(\sigma)$ $(-G^*(-\sigma))$ in the notation used previously in [4, Chapter 7, Section 4] has the form

$$J_\sigma(\sigma) = \tfrac{1}{2}\left\langle a^{-1}_{ijkl}(\mathbf{y})\sigma_{ij}\sigma_{kl}\right\rangle \tag{2.7}$$

where a^{-1}_{ijkl} is a tensor, inverse to a_{ijkl}, and by virtue of the results obtained previously [4, Chapter 7, Proposition 4.1 and Chapter 3, Theorem 4.1], the following relation is satisfied

$$\max_{\mathbf{u}\in V} J_u(\mathbf{u}) = \min_{\sigma_{ij}\in\Sigma} J_\sigma(\sigma) \tag{2.8}$$

Remark 2. To obtain (2.8) using the above-mentioned theorems from [4] it must be borne in mind that for the function $\mathbf{u} \in V$, which satisfies the condition $\langle \mathbf{u}\rangle = 0$, for the functional $J_u(\mathbf{u})$ the Korn inequality [6] is satisfied, in view of which the conditions of Theorem III. 4.1 [4] are satisfied on $V_0 = \{\mathbf{u} \in V : \langle \mathbf{u}\rangle = 0\}$, and by virtue of this equality, (2.8) holds on V_0. But since $J_u(u)$ cannot change its value when the body is displaced as a solid whole, relation (2.8) also holds for any $\mathbf{u} \in V$.

We will now consider formula (1.4). When $\alpha\beta = \gamma\delta$, $\nu = \mu = 1$ it takes the form

$$A^2_{\alpha\beta\alpha\delta} = \langle a_{\alpha\beta\alpha\beta}(\mathbf{y})y_3^2\rangle - 2J_u(\mathbf{u}) \tag{2.9}$$

Here we have taken relation (2.1) into account.

Cell problem (1.4) is Euler's equation for the problem of maximizing $J_u(u)$ on V. Its solution is unique on V_0 [6]. Then if we take Remark 2 into account, $\mathrm{N}^{\alpha\beta 1}$ solves the minimization problem given above. Bearing this in mind, from (2.7) and (2.9) we obtain

$$\left\langle a_{\alpha\beta\alpha\beta}(\mathbf{y})y_3^2\right\rangle - 2\max_{\mathbf{u}\in V} J_u(\mathbf{u}) = A^2_{\alpha\beta\alpha\beta}\left\langle a_{\alpha\beta\alpha\beta}(\mathbf{y})y_3^2\right\rangle - 2\min_{\sigma_{ij}\notin\Sigma} J_\sigma(\sigma) \tag{2.10}$$

which are two variational principle (in the strains and in the stresses) for the flexural stiffnesses of a non-homogeneous plate of periodic structure.

For arbitrary $\mathbf{u} \in V$, $\sigma_{ij} \in \Sigma$ from (2.11) we obtain an estimate of the upper and lower bounds for the stiffnesses

$$\left\langle a_{\alpha\beta\alpha\beta}(\mathbf{y})y_3^2\right\rangle - 2J_u(\mathbf{u}) \geqslant A^2_{\alpha\beta\alpha\beta} \geqslant \left\langle a_{\alpha\beta\alpha\beta}(\mathbf{y})y_3^2\right\rangle - 2J_\sigma(\sigma) \tag{2.11}$$

3. EXAMPLES

3.1. *The variational principle in stresses on a set that is independent of the elasticity constants.* Consider the variational principle in stresses. It is convenient to write it in terms of the quantities $\sigma'_{ij} = \sigma_{ij} - a_{ij\gamma\delta}(\mathbf{y})y_3$. Using Remark 1, we obtain from (2.10)

$$A^2_{\alpha\beta\alpha\beta} = \max_{\sigma_{ij}\in\Sigma'}\left\langle -a^{-1}_{ijkl}(\mathbf{y})\sigma'_{ij}\sigma'_{kl} + 2\sigma_{\alpha\beta}y_3\right\rangle \geqslant$$
$$\geqslant \left\langle a^{-1}_{ijkl}(\mathbf{y})\sigma'_{ij}\sigma'_{kl} + 2\sigma'_{\alpha\beta}y_3\right\rangle \text{ for any } \sigma'_{ij}\in\Sigma' \tag{3.1}$$

where

$$\Sigma' = \{\sigma_{ij} \in L(P_1) : \sigma_{ij,j} = 0 \text{ in } P_1,\ \sigma_{ij}n_j = 0 \text{ on } \Gamma_1,\ \sigma_{ij}n_j \text{ is periodic in } y_1, y_2 \in P_1\} \tag{3.2}$$

The set Σ' (3.2) describes the equilibrium stresses in P_1 with the corresponding boundary conditions. Here Σ' is independent of the elasticity constants. In particular, we can take as Σ' the stresses corresponding to the solutions of cell problem (1.4) with any elasticity constants, for example, solutions for a plate of homogeneous material.

3.2. *The upper limit.* Assuming $\mathbf{u} = 0$ in (2.11), we obtain

$$A^2_{\alpha\beta\alpha\beta} \le \left\langle a_{\alpha\beta\alpha\beta}(\mathbf{y})y_3^2 \right\rangle \tag{3.3}$$

3.3. *An estimate based on stresses in homogeneous plates with flat surfaces.* We will use relation (3.1) and the remark to it. In homogeneous plates with flat surfaces for bending $\sigma'_{\alpha\alpha} \ne 0$ ($\alpha\alpha$ = 11.22), while the remaining $\sigma'_{\alpha\beta} = 0$. Obviously, stresses of this form belong to Σ'. Consider the case when $\alpha = \beta = 1$—the flexural stiffness A^2_{1111}. The expression on the right-hand side of (3.1) for an isotropic material takes the form

$$\langle -\overset{-1}{a}_{1111}(y)\sigma'^2_{11} - \overset{-1}{a}_{2222}(y)\sigma'^2_{22} - 2\overset{-1}{a}_{1122}(y)\sigma'_{11}\sigma'_{22} + 2\sigma'_{11}y_3\rangle \tag{3.4}$$

Since the fact that they belong to Σ' does not impose any constraints on σ'_{11}, σ'_{22}, we can maximize expression (3.4) with respect to σ'_{11}, σ'_{22}. Euler's equation in this case has the form

$$-\overset{-1}{a}_{1111}\sigma'_{11} - \overset{-1}{a}_{1122}\sigma'_{22} + y_3 = 0$$
$$-\overset{-1}{a}_{1122}\sigma'_{11} - \overset{-1}{a}_{2222}\sigma'_{22} = 0$$

Taking into account the relations $\overset{-1}{a}_{1111} = \overset{-1}{a}_{2222} = 1/E$, $\overset{-1}{a}_{1122} = -\gamma/E$ (E and ν are Young's modulus and Poisson's ratio) we obtain for an isotropic material

$$-\frac{\sigma'_{11}}{E} + \frac{\nu\sigma'_{22}}{E} + y_3 = 0, \quad \frac{\nu\sigma'_{11}}{E} - \frac{\sigma_{22}}{E} = 0 \tag{3.5}$$

The solution of (3.5) is

$$\sigma'_{11} = F(\mathbf{y}), \ \sigma'_{22} = -\nu(\mathbf{y})F(\mathbf{y}); \ F(\mathbf{y}) = E(\mathbf{y})y_3/(1 - \nu^2(\mathbf{y}))$$

Note that this is the exact value of the stresses for homogeneous and multilayered plates [1, 7].
Substituting the stresses obtained into (4.2) we have

$$A^2_{1111} \ge \langle F(\mathbf{y})\rangle = \left\langle \frac{E(\mathbf{y})y_3^2}{1 - \nu^2(\mathbf{y})} \right\rangle \tag{3.6}$$

The right-hand side of this estimate corresponds to the classical formula for calculating the stiffnesses of homogeneous and multilayered plates. For other types of the plates (3.6) is an upper limit. Here $E(\mathbf{y})$ and $\gamma(\mathbf{y})$ can depend in an arbitrary way on $\mathbf{y}$. In particular, the estimate can be used for the case of perturbations of the layer geometry in a plate with a multilayered structure [8]. The fact that the stiffness of a multilayered plate with ideal layers gives minimum stiffness agrees with the fact that, in the corresponding formulae from [8], there is no linear term in the expansion of the stiffness in a series of perturbation theory.

Note that it follows from the fact that the right-hand side of (3.6) is identical with the stiffness of a plate with ideal layers, that a plate with non-flat layers will have a greater stiffness.

Thus, of the plates shown in Fig. 3, plate b has greater stiffness than plate a.

According to (3.3)

$$A^2_{1111} \le \left\langle \frac{E(\mathbf{y})(1 - \nu(\mathbf{y}))y_3^2}{(1 + \nu(\mathbf{y}))(1 - 2\nu(\mathbf{y}))} \right\rangle$$

The difference between the upper and lower limits when $\nu = 0.3$ for all components is $\sim 0.8\,E(y)y_3^2$.

3.4. *Unidirectional plates.* Suppose the region occupied by the plate has the form of a cylinder with directrix Oy_1 (Fig. 4). In this case $n_1 = 0$ on Γ_1—the free surface of the periodicity cell and stresses of the form $\sigma'_{11} \ne 0$, $\sigma'_{ij} = 0$ when $ij \ne 11$ belong to Σ' for any $\overset{1}{\sigma}_{11}$. Then, relation (3.1) for $\alpha = \beta = 1$ (the stiffness in the direction of the Oy_1 axis) takes the form

$$\langle -\overset{-1}{a}_{1111}(\mathbf{y})\sigma'^2_{11} + 2\sigma'_{11}y_3\rangle \tag{3.7}$$

It follows from Euler's equation for (3.7) that $\sigma'_{11} = -y_3/\overset{-1}{a}_{1111}$. Substituting this quantity into (3.7) and taking into account the fact that for isotropic materials $\overset{-1}{a}_{1111} = 1/E$, we obtain the estimate

$$A^2_{1111} \geqslant \langle E(y) y_3^2 \rangle \tag{3.8}$$

3.5. *An estimate of the stiffness in terms of the solution in the "inner" region.* Consider the stiffness to bending A^2_{2222} (in the Oy_2 direction) for a plate of the type shown in Fig. 4. It is clear from the mechanical point of view that in this direction the ribs have only a small effect on the bending, while a lower limit of the stiffness is given, for example, by the stiffness of the horizontal layer, shown hatched in Fig. 3. This can be shown rigorously from the limits in stresses.

Suppose $P \subset P_1$ is a subregion of the periodicity cell in which the following relations are satisfied

$$\sigma_{ij,j} = 0 \text{ in } P \tag{3.9}$$

$$\sigma_{ij} n_j = 0 \text{ on } S - \text{ the part of } \partial P, \text{ which do not intersect } \Gamma_0 \tag{3.10}$$

$$\sigma_{ij} n_j \text{ are periodic in } y_1, y_2 \in P_1 \tag{3.11}$$

We define

$$\sigma'_{ij} = \begin{cases} \sigma_{ij} \text{ in } P \\ 0 \text{ in } P_1 \setminus P \end{cases}$$

The function introduced $\sigma'_{ij} \in \Sigma'$. In fact, for any function $v \in D^\infty(P_1)$ (the set of infinitely differentiable finite functions on P_1 [9]) in view of (3.9), (3.10) and the formula for integration by parts, we have

$$\int_{P_1} \sigma'_{ij} v_{i,j} dy + \int_S \sigma_{ij} n_j v_i dy = 0 \tag{3.12}$$

where S is the part of the boundary of P defined in (3.10). The second integral in (3.12) is equal to zero, by virtue of which $\sigma'_{ij,j} = 0$ in P_1. The remaining conditions regarding belonging to Σ' are satisfied automatically.

As an example, consider a wafer plate, having two systems of ribs, directed along the Oy_1 and Oy_2 axes. A plate with any of these systems of ribs is imbedded in the initial one and is cylindrical. As a consequence of this, estimate (3.8) is satisfied for the stiffnesses of the wafer plate A^2_{1111} and A^2_{2222}.

3.6. *Random structures.* In bodies with a random structure, of the conditions introduced above, which are imposed on the permissible strains and stresses, one is not satisfied, namely, the periodicity condition. Consider a typical fragment of a random structure as the periodicity cell and repeated periodically. We obtain a plate with a periodic structure. From the mechanical point of view we would expect the initial plate and the plate obtained to be identical if the random distribution is spatially homogeneous and the chosen fragment is sufficiently large compared with the inhomogeneities (the "representative" fragment). The variational principles hold for a periodic medium. We

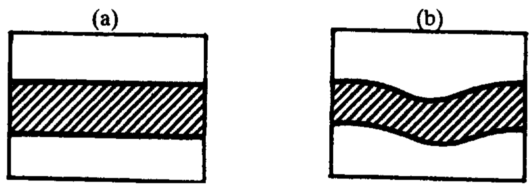

Fig. 3.

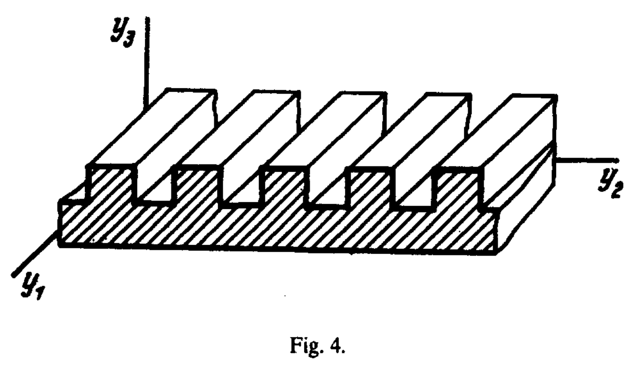

Fig. 4.

would wish to dispense with the periodicity condition. Such a method exists at the level of mechanical justification. It follows from the previous discussion that the local stresses must be periodic and they should, on average, be identical with the global stresses. But the latter are periodic (since they are simply constant) in the "representative" fragment. Hence, from the mechanical point of view the periodicity condition can be replaced by the condition that the mean value of the local stresses (or strains) should be identical with the global stresses (or strains), taking their type into account—global bending. Making this replacement we arrive at extension of the Nemat-Nasser-Hori variational principle [10] to the plate.

4. THE STIFFNESSES IN THE PLANE OF THE PLATE

Above we considered the case that is fundamental for plates, namely, the estimate of the flexural stiffnesses. Similar estimates can be obtained for the stiffnesses in the plane of the plate $A^0_{\alpha\beta\alpha\beta}$. To do this, we must put $\nu = \mu = 0$ in the calculations carried out above in Section 1, while in Section 2 we must omit the factor y_3 at the appropriate places. These actions lead to a variational principle of the form (2.10) but in this case

$$J_u(\mathbf{u}) = \langle a_{ij\gamma\delta}(\mathbf{y})u_{i,j}\rangle - \frac{1}{2}\langle a_{ijkl}(\mathbf{y})u_{i,j}u_{k,l}\rangle \tag{4.1}$$

$$\Sigma = \{\sigma_{ij} \in L(P_1) : (\sigma_{ij} - a_{ij\gamma\delta}(\mathbf{y}))_{,j} = 0 \text{ in } P_1,$$
$$(\sigma_{ij} - a_{ij\gamma\delta}(\mathbf{y}))n_j = 0 \text{ on } \Gamma_1,$$
$$\sigma_{ij}n_j \text{ are periodic in } y_1, y_2 \in P_1\}$$

Variational principle (3.1) takes the following form for the case considered

$$A^0_{\alpha\beta\alpha\beta} = \max_{\sigma'_{ij} \in \Sigma'} \langle -a^{-1}_{ijkl}(\mathbf{y})\sigma'_{ij}\sigma'_{kl} + 2\sigma_{\alpha\beta}\rangle \tag{4.2}$$

$J_\sigma(\sigma)$ and Σ' are the same as before.

The following estimates for the stiffnesses in the plane of the plate are analogues of the estimates obtained in Section 3

$$A^0_{\alpha\beta\alpha\beta} \leqslant \langle a_{\alpha\beta\alpha\beta}(\mathbf{y})\rangle \text{ for any type of plate}$$

$$A^0_{1111} \geqslant \left\langle \frac{E(\mathbf{y})}{1-\nu^2(\mathbf{y})} \right\rangle \text{ for plates with plane faces}$$

$$A^0_{1111} \geqslant \langle E(\mathbf{y})\rangle \text{ for unidirectional plates}$$

For plates with a random structure we also arrive in this case at a Nemat-Nasser-Hori type variational principle in respect of the strains in the plane of the plate.

5. MIXED (ASYMMETRICAL) STIFFNESSES

The stiffnesses $A^1_{\alpha\beta\alpha\beta}$ are "compensators" of the stiffnesses $A\nu{+}\mu\alpha\beta\alpha\beta$ ($\nu + \mu = 0.2$) when the local system of coordinates is changed in the following sense. The solution of the plate-theory problem does not depend on the "attachment" of the local system of coordinates (namely, the plane $y_3 = 0$) to the periodicity cell P_1, but $A^2_{\alpha\beta\alpha\beta}$ depend on it, see (1.2) and (1.3). This dependence is compensated by the change in $A^1_{\alpha\beta\alpha\beta}$. The specific form of this dependence was investigated elsewhere. Here it is important to consider directly the connectedness of $A^1_{\alpha\beta\alpha\beta}$ and $A^{\nu+\mu}_{\alpha\beta\alpha\beta}$ ($\nu + \mu = 0.2$) and the reference plane $y_3 = h$.

Consider the functional

$$J^h_u(\mathbf{u}) = \langle a_{ij\gamma\delta}(\mathbf{y})(y_3 + h)u_{i,j}\rangle - \frac{1}{2}\langle a_{ijkl}(\mathbf{y})u_{i,j}u_{k,l}\rangle \tag{5.1}$$

where h is an arbitrary non-zero number.

The solution of the problem

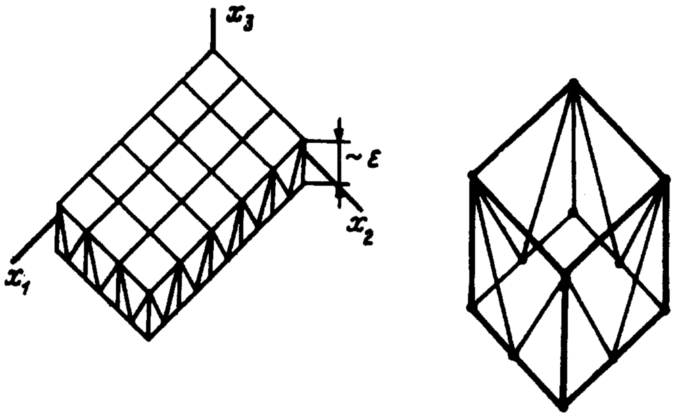

Fig. 5.

$$J_u^h(\mathbf{u}) \to \max, \quad \mathbf{u} \in V \tag{5.2}$$

as can easily be verified, is

$$\mathbf{N}^{\gamma\delta} = -\mathbf{N}^{\gamma\delta 1} + h\mathbf{N}^{\gamma\delta 0} \tag{5.3}$$

where $\mathbf{N}^{\gamma\delta\nu}$ ($\nu = 0, 1$) is the solution of cell problem (1.2).
Substituting (5.3) into (5.1) we obtain

$$\begin{aligned} J_u^h(\mathbf{u}) &= \langle a_{ij\gamma\delta}(\mathbf{y})(y_3 + h)(-N_{i,j}^{\gamma\delta 1} + hN_{i,j}^{\gamma\delta 0})\rangle - \frac{1}{2}\langle a_{ijkl}(\mathbf{y})(-N_{i,j}^{\gamma\delta 1} + hN_{i,j}^{\gamma\delta 0})(-N_{k,l}^{\gamma\delta 1} + hN_{k,l}^{\gamma\delta 0})\rangle \\ &= -\frac{1}{2}A_{\gamma\delta\gamma\delta}^2 - \frac{h^2}{2}A_{\gamma\delta\gamma\delta}^0 + \frac{1}{2}\langle a_{\gamma\delta\gamma\delta}(\mathbf{y})y_3^2\rangle + \frac{h^2}{2}\langle a_{\gamma\delta\gamma\delta}(\mathbf{y})\rangle - \\ &-\frac{1}{2}\langle -2a_{ij\gamma\delta}(\mathbf{y})N_{i,j}^{\gamma\delta 1} + 2a_{ij\gamma\delta}(\mathbf{y})y_3N_{i,j}^{\gamma\delta 0} - a_{ijkl}(\mathbf{y})N_{i,j}^{\gamma\delta 0}N_{k,l}^{\gamma\delta 1} + a_{ijkl}(\mathbf{y})N_{i,j}^{\gamma\delta 1}N_{k,l}^{\gamma\delta 0}\rangle \end{aligned} \tag{5.4}$$

Here we have used the equations (see Sections 2 and 4)

$$J_u^h(\mathbf{N}^{\gamma\delta 1}) = -\frac{1}{2}A_{\gamma\delta\gamma\delta}^2 + \frac{1}{2}\langle a_{\gamma\delta\gamma\delta}(\mathbf{y})y_3^2\rangle$$

$$J_u^h(\mathbf{N}^{\gamma\delta 0}) = -\frac{1}{2}A_{\gamma\delta\gamma\delta}^0 + \frac{1}{2}\langle a_{\gamma\delta\gamma\delta}(\mathbf{y})\rangle$$

the quantity $J_u(\mathbf{N}^{\gamma\delta\nu})$($\nu = 0, 1$) is given by (2.1) and (4.1), respectively.
From cell problem (1.2) we can derive the following equation (in the same was as (1.5) was derived)

$$\langle a_{ijkl}(\mathbf{y})(N_{k,l}^{\gamma\delta\nu} + (-1)^\nu y_3^\nu \delta_{k\gamma}\delta_{l\delta})N_{i,j}^{\alpha\beta\mu}\rangle = 0 \tag{5.5}$$

Using (5.5) and taking into account the fact that, by definition (1.3)

$$A_{\gamma\delta\gamma\delta}^1 = \langle a_{\gamma\delta\gamma\delta}(\mathbf{y})y_3 - a_{\gamma\delta kl}(\mathbf{y})N_{k,l}^{\gamma\delta 1}\rangle = \langle y_3(a_{\gamma\delta\gamma\delta}(\mathbf{y}) + a_{\gamma\delta kl}(\mathbf{y})N_{k,l}^{\gamma\delta 0})\rangle$$

we obtain

$$J_u^h(\mathbf{N}^{\gamma\delta}) = -\frac{1}{2}A_{\gamma\delta\gamma\delta}^2 - \frac{h^2}{2}A_{\gamma\delta\gamma\delta}^0 + \frac{1}{2}\langle a_{\gamma\delta\gamma\delta}(\mathbf{y})y_3^2\rangle + \frac{h^2}{2}\langle a_{\gamma\delta\gamma\delta}(\mathbf{y})\rangle - hA_{\gamma\delta\gamma\delta}^1 + h\langle a_{\gamma\delta\gamma\delta}(\mathbf{y})y_3\rangle \tag{5.6}$$

Taking into account the definition of $\mathbf{N}^{\gamma\delta}$ as the solution of problem (5.2) we obtain the following variational principle from (5.6)

$$\langle a_{\gamma\delta\gamma\delta}(\mathbf{y})y_3^2\rangle + 2h\langle a_{\gamma\delta\gamma\delta}(\mathbf{y})y_3\rangle + h^2\langle a_{\gamma\delta\gamma\delta}(\mathbf{y})\rangle - (A_{\gamma\delta\gamma\delta}^2 + 2hA_{\gamma\delta\gamma\delta}^1 + h^2A_{\gamma\delta\gamma\delta}^0) = \max_{\mathbf{u}\in V} J_u^h(\mathbf{u}) \tag{5.7}$$

The quantity on the right-hand side of (5.7) is equal to

$$\min_{\sigma_{ij}\in\Sigma_h} J_\sigma(\sigma) \tag{5.8}$$

where Σ_h is obtained by replacing y_3 by $y_3 + h$ in (2.6). Taking (5.8) into account we obtain the dual variational principle (in stresses).

In this section we have obtained variational principles for the combination of stiffnesses $A^{\nu+\mu}_{\gamma\delta\gamma\delta}$ ($\nu + \mu = 0.1$) given on the left-hand side of (5.7). This corresponds to what was said at the beginning of this section. Since the stiffnesses of a plate in a plane and on bending are calculated (estimated) independently, (5.7) and (5.8) are the variational principle for mixed (asymmetrical) stiffnesses.

6. APPLICATION OF VARIATIONAL PRINCIPLES TO FINITE-DIMENSIONAL STRUCTURES

The examples given in Section 3 refer to one of a number of types of plates. In this section we will consider one other form of plate, for which the use of variational principles is promising, namely, a plate of complex structure [11, 12], the periodicity cell of which has a finite-dimensional construction.

As an example we will consider a structure, elongated in the Ox_1x_2 plane, the periodicity cell of which is composed of rods (of the type shown in Fig. 5). This structure includes various types of open coverings (in the terminology generally employed [13], they can be characterized as "open meshes").

The displacements of the nodes of the periodicity cell $\mathbf{U}$ are determined by the strain energy of the rods, in view of which the functional $J_u(\mathbf{u})$ is expressed in terms of $\mathbf{U}$

$$J_u(\mathbf{u}) = J(\mathbf{U}) \tag{6.1}$$

If the elements of the periodicity cell are beams (i.e. their flexures play an important role), the introduction of generalized displacements V (including rotation of the nodes) also leads to an expression of the form (6.1).

Minimizing (6.1) we obtain the stiffnesses of the open mesh.

Practical methods of carrying out a finite-dimensional minimization are well developed [14]. The advantage of the proposed approach is that it is possible to avoid deriving the equilibrium equations, which is a non-trivial problem even for single-layer meshes [11].

In the general case, in particular for anisotropic plates, the effectiveness of the proposed method is similar to that of traditional variational methods [14]. Whether effective estimates can be obtained depends on a successful choice of the permissible strain or stress fields. In any case, we use the finite-elements method. The case of rod (beam) structures discussed above is an example of the use of the finite-elements method.

REFERENCES

1. ANNIN, B. D., KALAMKAROV, A. L., KOLPAKOV, A. G. and PARTON, V. Z., *Analysis and Design of Composites and Structural Elements*. Nauka, Novosibirsk, 1993.
2. CAILLERIE, D., Thin elastic and periodic plates. *Math. Meth. in Appl. Sci.*, 1984, **6**, 2, 159–191.
3. RABOTNOV, Yu. N., *Solid Mechanics of Deformable Solid Bodies*. Nauka, Moscow, 1979.
4. EKELAND, I. and TEMAM, R., *Convex Analysis and Variational Problems*. North-Holland, Amsterdam; American Elsevier, New York, 1976.
5. KOLPAKOV, A. G., Variational principles for stiffnesses of a non-homogeneous beam. *J. Mech. Phys.* Solids., 1988, **46**, 1039–1053.
6. BAKHALOV, N. S. and PANASENKO, G. P., *Process Averaging in Periodic Media*. Nauka, Moscow, 1984.
7. KOLPAKOV, A. G., Analysis and design of multilayered plates. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1989, 4, 152–161.
8. KILPAKOV, A. G., Structural sensitivity of multilayered composites. *Izv. Ross. Akad. Nauk. MTT*, 1993, 6, 30–37.
9. LIONS, J.-L. and MAGENES, E., *Problèmes aux limites non-homogenes et applications*. Dunod, Paris, 1968.
10. NEMAT-NASSER, S. and HORI, M. *Micromechanics: Overall Properties of Heterogeneous Materials.* Elsevier, Amsterdam, 1993, 687.
11. KALAMKAROV, A. L. and KOLPAKOV, A. G., *Analysis, Design and Optimization of Composite Structures*. John Wiley, Chichester, 1977, 356.
12. KALAMKAROV, A. L. and KOLPAKOV, A. G., On the analysis and design of fiber-reinforced composite shells. *Trans. ASME J. Appl. Mech.*, 1996, **63**, 939–945.
13. PSHENICHNOV, G. I., *The Theory of Thin Elastic Lattice Shells and Plates*. Nauka, Moscow, 1982.
14. WASHIZU, K., *Variational Methods in Elasticity and Plasticity*. Pergamon, New York, 1968.

Translated by R.C.G.